ELSEVIER

Journal of Pure and Applied Algebra 97 (1994) 91–107

JOURNAL OF PURE AND APPLIED ALGEBRA

Dimension in Bredon cohomology

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Communicated by A. Heller; received 23 October 1991; revised 4 May 1992

Abstract

We define and investigate dimension in Bredon cohomology of G-CW-complexes, where G is a finite group. We obtain results which in the case of trivial coefficients describe relations between vanishing of the cohomology groups of fixed points sets and orbit spaes of the induced action of appropriate subgroups of G.

1. Introduction

Let G be a finite group. In this paper we will study the equivariant cohomology theories defined on the category G-CW of G-CW-complexes by Bredon in [1]. By a G-CW-complex we mean a CW-complex K with a given cellular action of G such that, for every subgroup H of G, the fixed point set K^H is a subcomplex of K. It follows from this definition that G acts on the set SK of all open cells of K and that, for every $s \in SK$ and $k \in s$, the isotropy groups $G_k = \{g \in G \mid gk = k\}$ and G_s, are equal.

The main results of this paper, Theorems 1.4–1.6, specialize to the following results. Assume that A is an abelian group, m is a natural number, G is a finite p-group and K is a G-CW-complex. Let

$$F(K) = \{G_k \mid k \in K\} \quad \text{and} \quad E_G(K) = \bigcup_{H \in F(K)} E_G(H),$$

where $E_G(H)$ is the family of all subgroups J of G such that $J \cap H$ is a normal subgroup of J and $J/J \cap H$ is an elementary abelian p-group.

Theorem 1.1. *Suppose that, for every subgroup H of G and $n \geq m$,*

$$H^n(K/H, A) = 0.$$

Then, for every subgroup H of G and $n \geq m$,

$$H^n(K^H, A) = H^n(K^H/NH, A) = 0.$$

where NH is the normalizer of H.

SSDI 0022-4049(93)E0173-2

Theorem 1.2. *Assume that, for every* $E \in E_G(K)$ *and* $n \geq m$,

$$H^n(K/E, A) = 0.$$

Then, for every subgroup H *of* G *and* $n \geq m$,

$$H^n(K/H, A) = 0.$$

Theorem 1.3. *Assume that, for every subgroup* H *of* G *and* $n \geq m$,

$$H^n(K^H, A) = 0$$

Suppose that there is a natural number $q \geq m + 4\log_p|G|$ *such that*

$$H^q(K/E, A) = H^{q+1}(K/E, A) = 0$$

whenever $E \in E_G(K)$. *Then, for* $n \geq m$,

$$H^n(K/G, A) = 0.$$

In the case where $K = EG$ is the universal free G-complex, Theorems 1.2 and specialize to the well-known results of the cohomology theory of groups [2, 4].

In order to state our main results we recall briefly the definition of the Bred cohomology. We will assume that G is a finite group. Let O_G be the category canonical G-orbits defined in [1]. Its objects are the G-sets of the form G/H, wh H is a subgroup of G. The morphisms of O_G are the equivariant maps. Coefficients Bredon cohomology are contravariant functors from O_G to the category Ab of abeli groups. Let $C_*(-, \mathbb{Z})$ denote the cellular chain complex functor from the category C to the category Ab_* of chain complexes in Ab. Assume that K is a G-CW-complex. L

$$C_*(K): O_G \to \text{Ab}_*$$

be the contravariant functor given by $C_*(K)(G/H) = C_*(K^H, \mathbb{Z})$ on objects, and the induced maps on morphisms. The nth Bredon cohomology group $H^n(K, M)$ then the nth cohomology group of the cochain complex

$$\text{Hom}_{O_G}(C_*(K), M),$$

where $\text{Hom}_{O_G}(-,-)$ denotes the abelian group of all natural transformations of cc travariant functors from O_G to Ab.

Let H be a subgroup of G and let K' be an H-subcomplex of K. By

$$K \times_{K/G} K'/H$$

we will denote the G-CW-complex equal to the fibre product (pull-back) of t natural projections $K \to K/G$ and $K'/H \to K/G$ to the orbit space. The group G a on this G-CW-complex by the action on the first coordinate. If M is a G-coeffici system and $K' = K$, then it follows from the classical "uniqueness theorem" in [1, IV.5], that

$$H^n_G(K \times_{K/G} K/H, M) = H^n_G(K, M[G/H]),$$

where $M[G/H]$ is the G-coefficient system such that

$$M[G/H](-) = M(-) \otimes \mathrm{Hom}_{\mathbb{Z}(G)}(\mathbb{Z}(-), \mathbb{Z}(G/H)).$$

Here $\mathbb{Z}(G/H)$ denotes the free permutation module with the basis G/H.

We can now state our main results. We assume that m is a natural number, G is a finite group, K is a G-CW-complex and M is a G-coefficient system.

Theorem 1.4. *Let H be a p-subgroup of G such that, for every subgroup J of H and $n \geq m$,*

$$H^n_G(K \times_{K/G} K/J, M) = 0.$$

Then, for every subgroup J of H and $n \geq m$,

$$H^n_G(K \times_{K/G} K^J, M) = 0 = H^n_G(K \times_{K/G} K^J/H \cap NJ, M).$$

Let $\mathscr{S}^p_G$ denote the set of all Sylow p-subgroups of G, and let

$$\mathscr{S}_G = \bigcup_{p \in P_G} \mathscr{S}^p_G,$$

where P_G is the set of all primes p dividing $|G|$. We will use the notation

$$E_G(K) = \bigcup_{H \in \mathscr{S}_G} E_H(K).$$

It follows from this definition that a p-subgroup J of G belongs to $E_G(K)$ if and only if there exists $k \in K$ such that J_k is a normal subgroup of J and J/J_k is an elementary abelian p-group.

Theorem 1.5. *Assume that, for every $E \in E_G(K)$ and $n \geq m$,*

$$H^n_G(K \times_{K/G} K/E, M) = 0.$$

Then, for every subgroup H of G and $n \geq m$,

$$H^n_G(K \times_{K/G} K/H, M) = 0.$$

Theorem 1.6. *Assume that*

$$H^n_G(K \times_{K/G} K^E, M) = 0$$

whenever $E \in E_G(K)$ *and* $n \geq m$. *Let* q *be a natural number such that, for every* $p \in P_G$ *and* $G_p \in \mathscr{S}^p_G$,

$$q \geq m + 4\log_p|G_p|.$$

Suppose that, for every $E \in E_G(K)$,

$$H^q_G(K \times_{K/G} K/E, M) = H^{q+1}_G(K \times_{K/G} K/E, M) = 0.$$

Then, for every subgroup H *of* G *and every* $n \geq m$

$$H^n_G(K \times_{K/G} K/H, M) = 0.$$

In Section 2 we will recall the definition of cohomological dimension in Bredon cohomology (introduced in [7]) and reformulate the statements of the above theorems in terms of this dimension (Theorems 2.7–2.9). We will also show there how to obtain Theorems 1.4–1.6 from Theorems 2.7–2.9. Theorems 2.7 through 2.9 will be proved in Section 3. In their proofs we will use the main technical results (Theorems 1.5 and 1.6) of [6] and some of the results of [7].

Let

$$K^{>H} = \{k \in K \mid H_k \neq H\}.$$

In [6] we proved the following fact (Theorem 1.1 of [6]).

Theorem 1.7. *Assume that, for every subgroup* H *of* G *and every* $n \geq m$,

$$H^n_G(K \times_{K/G} (K^H, K^{>H}); M) = 0.$$

Let G' *be a subgroup of* G *such that, for every prime* p *and every* p*-subgroup* H *of* G,

$$H^n_G(K \times_{K/G} K/H, M) = 0$$

whenever $m \leq n \leq n + 1 + 4\log_p|G'_p|$ *where* G'_p *is a Sylow* p*-subgroup of* G'. *Then, for all* $n \geq m$,

$$H^n_G(K \times_{K/G} K/G', M) = 0. \qquad \square$$

If $G' = G$, then Theorem 1.7 is a consequence of Theorem 1.6. In fact one can state and prove Theorem 1.6 in a stronger form which implies Theorem 1.7. Theorem 1.7 will not be used in our proofs of Theorems 1.4–1.6.

We now show how to obtain Theorems 1.1–1.3 from Theorems 1.4–1.6. Let A be an abelian group. By the same symbol A we will denote the constant contravariant functor from O_G to Ab. It is easy to check that, for every subgroup H of G,

$$H^n_G(K \times_{K/G} K'/H, A) = H^n(K'/H, A).$$

Thus Theorem 1.1 is a consequence of Theorem 1.4, Theorem 1.2 is a consequence of Theorem 1.5 and Theorem 1.3 is a consequence of Theorem 1.6.

2. Preliminaries

In this section we generalize slightly the definition of Bredon cohomology groups. These cohomology groups will be used later in proofs of our main results. We also outlined their properties as described in [5], [6], and [7].

Let G be a finite group let H be a subgroup of G. By $O_{G,H}$ we will denote the category whose objects are the canonical G-orbits and whose morphisms are the G-maps $f_{[h]}: G/J \to G/J'$ determined by the elements h of H satisfying the condition $h^{-1}Jh \subseteq J'$. Here, $f_{[h]}(gJ) = ghJ'$ for $g \in G$. There is a split surjection of categories $\pi: O_{G,H} \to O_H$ given by sending G/J to $H/H \cap J$ and a morphism f to its restriction. A right inverse $i: O_H \to O_{G,H}$ is given by $i(H/L) = G \times_H (H/L)$ and $i(f) = G \times_H f$.

Assume now that K is a G-KW-complex and that K' is an H-subcomplex of K. For every subgroup J of G let

$$K'^J = \{k \in K' \mid J \subseteq G_k\}.$$

Let

$$c_*(K'): O_{G,H} \to \mathrm{Ab}$$

be the contravariant functor such that, for every $J \subseteq G$,

$$c_*(K')(G/J) = C_*(K'^J, \mathbb{Z})$$

and, for every morphism $f_{[h]}$ of $O_{G,H}$,

$$c_*(K')(f_{[h]}) = C_*([h], \mathbb{Z}),$$

where $[h]: K'^{J'} \to K'^J$ is induced by the operation by h.

If M is a contravariant functor from $O_{G,H}$ to Ab, then we define $H^n_{\{H\}}(K', M)$ to be equal to the nth cohomology group of the cochain complex

$$\mathrm{Hom}_{O_{G,H}}(c_*(K'), M).$$

This cohomology groups are the Bredon cohomology groups of the H-CW-complex K' with the local H-coefficient system $\mathscr{L}$. If s is an open cell of K' and $K(s)$ is the smallest CW-subcomplex of K' which contains s, then

$$\mathscr{L}(K(s)) = M(G/G_s).$$

If, for every $k \in K'$, $H_k = G_k$, then

$$H^n_{\{H\}}(K', M) = H^n_H(K', Mi).$$

If $M = M'\pi$ where $M' : O_H \to \mathrm{Ab}$, then

$$H^n_{\{H\}}(K', M) = H^n_H(K', M').$$

In particular, if $M = A$ is a constant contravariant functor, then

$$H^n_{\{H\}}(K', A) = H^n(K'/H, A).$$

If $K = G/J = K'$, then

$$H^n_{\{H\}}(G/J, M) = \prod_{[g] \in H \backslash G/J} M(G/gJg^{-1}).$$

It is obvious that if $H = G$, then $O_{G,G} = O_G$ and $H^*_{\{G\}}(-,-) = H^*_G(-,-)$.

Let F be a set of subgroups of G. By $O_{F,H}$ we will denote the full subcategory $O_{G,H}$ whose objects are canonical G-orbits G/J such that $J \in F$. Let

$$F(K') = \{G_k \mid k \in K'\}.$$

Then

$$\mathrm{Hom}_{O_{G,H}}(c_*(K'), M) = \mathrm{Hom}_{O_{F(K'),H}}(c_*(K'), M).$$

Assume that J is a subgroup of H. Let

$$\iota_J : O_{G,J} \to O_{G,H}$$

be the natural inclusion of categories. By $H^*_{\{J\}}(K', M)$ we will denote the cohomolc groups of the cochain complex

$$\mathrm{Hom}_{O_{G,J}}(c_*(K')\iota_J, M\iota_J).$$

Let

$$M[H/J] : O_{G,H} \to \mathrm{Ab}$$

be the contravariant functor such that, for $L \subseteq G$,

$$M[H/J](G/L) = M(G/L) \otimes \mathrm{Hom}_{\mathbb{Z}(H)}(\mathbb{Z}(H/H \cap L), \mathbb{Z}(H/J))$$

and for $f_{[h]} : G/L \to G/L'$,

$$M[H/J](f_{[h]}) = M(f_{[h]}) \otimes \mathrm{Hom}_{\mathbb{Z}(H)}(\mathbb{Z}(f'_{[h]}), \mathrm{id}),$$

where $f'_{[h]} : H/H \cap L \to H/H \cap L'$ is the H-map determined by h. It is proved in [that

$$H^*_{\{J\}}(K', M) = H^*_{\{H\}}(K', M[H/J])$$

and that, if $G = H$, then

$$H^*_{\{J\}}(K', M) = H^*_G(K \times_{K/G} K'/J, M).$$

Let e be the neutral element of G. It follows from the definition that $O_{G,e}$ is the poset of all subgroups of G. There exists a natural H-action on the complex

$$\mathrm{Hom}_{O_{G,e}}(c_*(K')\imath_e, M\imath_e)$$

such that, for every subgroup J of H,

$$\mathrm{Hom}_{O_{G,J}}(c_*(K')\imath_J, M\imath_J) = \mathrm{Hom}_{O_{G,e}}(c_*(K')\imath_e, M\imath_e)^J.$$

The following facts are a generalization of the well-known results of the cohomology theory of groups and can be proved by similar methods to those described in [2, III.10].

Proposition 2.1. (i) *Suppose that J is a subgroup of H such that $|H/J|$ is a power of p. If $H^n_{\{J\}}(K,M)$ is a p-group, then $H^n_{\{H\}}(K,M)$ is also a p-group.*

(ii) *Suppose that*

$$H^n_{\{e\}}(K,M) = 0.$$

Then

$$|H|\,H^n_{\{H\}}(K,M) = 0.$$

If, additionally, for every prime p dividing $|H|$,

$$H^n_{\{H_p\}}(K,M) = 0,$$

where H_p is a Sylow p-subgroup of H, then

$$H^n_{\{H\}}(K,M) = 0. \qquad \square$$

In [7], we have proved the following result.

Proposition 2.2. *Let m and q be natural numbers such that $q < m$. Let H be a p-subgroup of G. Suppose that J is a subgroup of H such that*

$$H^n_{\{J\}}(K,M) = 0$$

whenever $q \le n \le m$. Let

$$H^n_{\{J'\}}(K,M) = 0$$

whenever $J \subseteq J' \subseteq H$ and $n = m, m-1$. Then

$$H^n_{\{H\}}(K,M) = 0$$

whenever $q \le n \le m$. $\square$

The statements (i) and (ii) of the next results are proved in [6]. The assertion (iii) is proved in [7].

Proposition 2.3. *Assume that H is a p-subgroup of G and that J is a subgroup of H. Suppose that K'' is an H-subcomplex of K' such that, for every point $k \in K' \backslash K''$, H_k is a subgroup of J. Let m be a natural number.*

(i) *Suppose that, for $n = m, m+1$, $H^n_{\{J\}}(K', K''; M)$ is a p-group and*

$$H^n_{\{H\}}(K', K''; M) = 0.$$

Then, for $n = m, m+1$,

$$H^n_{\{J\}}(K', K''; M) = 0.$$

(ii) *Suppose that, for $n = m, m+1, m+2$,*

$$H^n_{\{J\}}(K', K''; M) = 0$$

and, for $n = m, m+1$,

$$H^n_{\{H\}}(K', K''; M) = 0.$$

Then

$$H^{m+2}_{\{H\}}(K', K''; M) = 0.$$

(iii) *Let J be a normal subgroup of H. Assume that*

$$H^n_{\{J'\}}(K', K''; M) = 0$$

whenever $n \geq m$, $J \subseteq J' \subseteq H$, and J'/J is an elementary abelian p-group. Then, for all $n \geq m$,

$$H^n_{\{H\}}(K', K''; M) = 0. \qquad \square$$

We recall now the definition of the dimension in Bredon cohomology, which was introduced in [7]. Let H be a subgroup of G and let $M : O_{G,H} \to \mathrm{Ab}$ be a coefficient system. Assume that K is a G-CW-complex and that K' is an H-subcomplex of K. Let $\mathbb{N}^+ = \mathbb{N} \cup \{\infty\}$, where $\mathbb{N}$ is the set of all natural numbers. We define $\mathrm{cd}_H(K', M)$ to be the smallest element d of $\mathbb{N}^+$ such that, for every $n \geq d + 1$,

$$H^n_{\{H\}}(K', M) = 0.$$

If F is a set of subgroups of H, then

$$\mathrm{cd}_F(K', M) = \max\{\mathrm{cd}_J(K', M) \mid J \in F\}.$$

The number

$$\mathrm{cd}(H, K', M) = \max\{\mathrm{cd}_J(K', M) \mid J \subseteq H\}$$

will be called the cohomological dimension of the H-subcomplex K' of K with coefficients in M. We will also consider the natural extension of the above definitions to the case of a pair of H-subcomplexes of K.

As immediate consequences of Propositions 2.1 and 2.2 we obtain the following facts.

Corollary 2.4. (i) *For any subgroup H of G,*

$$\mathrm{cd}(H, K', M) = \max\{\mathrm{cd}(H_p, K', M) \mid p \in P_G,\ H_p \in \mathscr{S}_H^p\}.$$

(ii) *If* $\mathrm{cd}(H, K', M)$ *is a finite number, then*

$$\mathrm{cd}(H, K', M) = \mathrm{cd}_e(K', M). \qquad \square$$

We will use the following notation. Let J be a subgroup of H. Then

$$F(J, H) = \{J' \mid J \subseteq J' \subseteq H\}.$$

If J is a normal subgroup of a p-group H, then $E(J, H)$ will denote the subset of $F(J, H)$ such that $J' \in E(J, H)$ if and only if J'/J is an elementary abelian p-group.

Corollary 2.5. *Let H be a p-subgroup of G and let J be a subgroup of H. If*

$$\mathrm{cd}_{F(J,H)}(K', M) < \infty,$$

then

$$\mathrm{cd}_{F(J,H)}(K', M) = \mathrm{cd}_J(K', M).$$

Proof. This result is a consequence of Proposition 2.2. $\square$

Proposition 2.6. *Let H be a p-subgroup of G and let J be a subgroup of H. Suppose that (K', K'') is a pair of H-CW-complexes of K such that, for every $k \in K' \backslash K''$, $H_k \subseteq J$. Then the following equalities hold.*

(i) $\mathrm{cd}_{F(J, H \cap NJ)}(K', K''; M) = \mathrm{cd}_{E(J, H \cap NJ)}(K', K''; M)$.

(ii) $\mathrm{cd}_{F(J,H)}(K', M) = \max(\mathrm{cd}_J(K', K''; M), \mathrm{cd}_H(K', K''; M))$.

(iii) *If there is a natural number $m > \mathrm{cd}_J(K', K''; M)$ such that, for $n = m, m+1$,*

$$H^n_{\{H\}}(K', K''; M) = 0,$$

then

$$\mathrm{cd}_{F(J,H)}(K', K''; M) = \mathrm{cd}_J(K', K''; M).$$

Proof. The equality (i) is a consequence of Proposition 2.3(iii). The equality (ii) follows from Propositions 2.1 and 2.3(i). The assertion (iii) can be obtained from Propositions 2.3(ii) and 2.6(ii) and Corollary 2.5. □

We now state three theorems which imply Theorems 1.4–1.6. They will be proved in Section 3 of this paper.

Theorem 2.7. *Let H be a p-subgroup of G and let $M : O_{G,H} \to \mathrm{Ab}$ be a coefficient system. Let m be a natural number such that, for every $J \subseteq H$,*

$$\mathrm{cd}_e(K^J, M) < m.$$

Assume that there exists a natural number $q \geq m + 4\log_p|H|$ such that, for every $J \subseteq H$,

$$H^q_{\{J\}}(K, M) = 0 = H^{q+1}_{\{J\}}(K, M).$$

Then

$$\mathrm{cd}(H, K, M) < m.$$

Theorem 2.8. *Let H be a p-subgroup of G and let $M : O_{G,H} \to \mathrm{Ab}$ be a coefficient system. Then, for any subgroup J of H,*

$$\mathrm{cd}(H \cap NJ, K^J, M) \leq \mathrm{cd}(H, K, M).$$

Theorem 2.9. *If G is a finite group, and $M : O_G \to \mathrm{Ab}$ is a G-coefficient system, then*

$$\mathrm{cd}(G, K, M) = \mathrm{cd}_{E_G(K)}(K, M).$$

We will now prove Theorems 1.4–1.6 using the theorems above. Let us recall that if $M : O_G \to \mathrm{Ab}$ is a G-coefficient system then

$$H^n_{\{J\}}(K, M) = H^n_G(K, M[G/J]) = H^n_G(K \times_{K/G} K/J, M).$$

Thus Theorem 1.4 is an immediate consequence of Theorem 2.8 and Theorem 1.5 is a consequence of Theorem 2.9. From the assumptions of Theorem 1.6, and from Theorem 2.7, it follows that, for every $E \in E_G(K)$, $\mathrm{cd}(E, K, M) \leq m - 1$. Now, it is sufficient to apply Theorem 2.9 to obtain Theorem 1.6.

3. Main results

In our proofs of the main results of this paper we will consider certain pairs of CW-subcomplexes of K indexed by appropriate triples of subgroups of G. These pairs were introduced in [6]. We recall now the definition.

Suppose that J, H and L are subgroups of G such that $J \subseteq H \subseteq L$ and J is a normal subgroup of H. We will consider the following sets of subgroups:

$$F_1 = \{U \subseteq H \mid UJ = H\}$$

and

$$F_0 = \{U \subseteq L \mid UJ \neq H = (U \cap H)J\}.$$

Let, for $i = 0,1$, $K_i = \bigcup_{U \in F_i} K^U$. We will denote the pair (K_1, K_0) of CW-subcomplexes of K by $K(J,H,L)$. If $J = e$, then $K_1 = K^H$ and

$$K_1 \backslash K_0 = \{k \in K \mid L_k = H\}.$$

If $J = H$, then $K_1 = K$ and

$$K_1 \backslash K_0 = \{k \in K \mid L_k \subseteq H\}.$$

If $H = L$, then $K_0 = \emptyset$. Hence

$$K(H,H,H) = (K, \emptyset), \qquad K(e,H,H) = (K^H, \emptyset).$$

If G' is a subgroup of $NJ \cap NH \cap NL$, then $K(J,H,L)$ is a pair of G'-subcomplexes of K. It follows from the definition that, if $G' \subseteq L \cap NH$, then

$$H^*_{\{G'\}}(K(e,H,L); M) = H^*_{\{G'H\}}(K(e,H,L); M).$$

Let H', G' be p-subgroups of G such that $H' \subseteq G'$. By $P(H',G')$ we will denote the set of all pairs of CW-subcomplexes of K of the form $K(J,H,L)$ where $J \subseteq H \subseteq H' \subseteq L \subseteq G' \cap NJ$ and H'/J is an elementary abelian p-group. (In [6] this set was denoted by $C_K(H',G')$.) The following fact is a consequence of the main technical results, Theorems 1.5 and 1.6, of [6]. It will be used in the proof of Theorems 2.7–2.9.

Proposition 3.1. *Suppose that H is a p-subgroup of G such that, for every subgroup J of H,*

$$H^n_{\{J\}}(K, M) = 0$$

whenever $m \leq n \leq 4\log_p|H| + 1$. Assume that

$$H^n_{\{e\}}(K_1, K_0; M) = 0$$

whenever $J \subseteq J' \subseteq H$, $(K_1, K_0) \in P(J, J')$ and $n \geq m + \log_p|J'|$.

Then

$$H^n_{\{J\}}(K_1, K_0; M) = 0$$

whenever $n \geq m + \log_p|H|$, $J \subseteq H$ *and* $(K_1, K_0) \in P(J, H)$. $\square$

The following lemma is easy to prove, and can also be obtained from the results of [6, Section 2].

Lemma 3.2. (i) *Let* L *be a p-subgroup of* G *and let* H, J *and* J' *be subgroups of* L *such that* $H \subseteq NJ' \cap NJ$, $J \subseteq J' \subseteq NJ$. *Suppose that, for every subgroup* J'' *of* J',

$$H^m_{\{H \cap NJ''\}}(K(e, J'', L); M) = 0.$$

Then

$$H^m_{\{H\}}(K(J, J', L); M) = 0.$$

(ii) *Let* H *be a p-subgroup of* G *and let* m *be a natural number. Assume that, for every* $J \subseteq H$ *and every* $n \geq m$,

$$H^n_{\{e\}}(K^J, M) = 0.$$

Then

$$H^n_{\{e\}}(K_1, K_0; M) = 0$$

whenever $(K_1, K_0) \in P(J, H)$, $J \subseteq H$ *and* $n \geq m + \log_p|H|$. $\square$

We can now prove one of the results stated in Section 2.

Proof of Theorem 2.7. Suppose that the assumptions of Theorem 2.7 hold. Then Lemma 3.2(ii) implies that $H^n_{\{e\}}(K_1, K_0; M) = 0$ whenever $J \subseteq J' \subseteq H$, $(K_1, K_0) \in P(J, J')$, and $n \geq m + \log_p|J'|$. From Proposition 2.2 we obtain that $H^n_{\{J\}}(K, M) = 0$ whenever $J \subseteq H$ and $m \leq n \leq m + 1 + 4\log_p|H|$. Proposition 3.1 implies now that $H^n_{\{J\}}(K, M) = 0$ whenever $J \subseteq H$ and $n \geq m + \log_p|J|$. Thus $\operatorname{cd}(H, K, M) < \infty$ and from Corollary 2.4(ii) it follows that this number is equal to $\operatorname{cd}_e(K, M)$ so is smaller than m. $\square$

In our proof of Theorem 2.8 we will use the following fact.

Proposition 3.3. *Let* H *be a p-subgroup of* G. *Then, for any subgroup* J *of* H,

$$\operatorname{cd}_e(K^J, M) \leq \operatorname{cd}(H, K, M).$$

The proposition above will be proved at the end of the paper.

Proof of Theorem 2.8. Assume that $\mathrm{cd}(H, K, M) = m - 1$ is a finite number. Then Proposition 3.3 implies that, for $J \subseteq H$ and $n \geq m$, $H^n_{\{e\}}(K^J, M) = 0$. Proposition 3.1 and Lemma 3.2 imply that

$$H^n_{\{H'\}}(K(J, J, H \cap NJ); M) = 0$$

whenever $n \geq m + \log_p|H|$ and H' is a subgroup of H such that J is a normal subgroup of H' and H'/J is an abelian elementary p-group. From Proposition 2.6 we obtain that, for every $J \subseteq H$,

$$\mathrm{cd}_{H \cap NJ}(K(J, J, H \cap NJ); M) \leq m - 1 + \log_p|H|.$$

We will now prove, by the induction on $|H|$, that, for every $J \subseteq H$ and $n \geq m + \log_p|H|$,

$$H^n_{\{H \cap NJ\}}(K(e, J, H); M) = 0.$$

It is easy to check that

$$K(e, J, H) = K(e, J, H \cap NJ).$$

Thus it is sufficient to consider only the case where J is a normal subgroup of H. In this case we will proceed by the induction on $|J|$. It follows from the inductive assumptions that

$$H^n_{\{H \cap NJ'\}}(K(e, J', H); M) = 0$$

whenever $J' \subseteq J$, $J' \neq J$, and $n \geq m + \log_p|H|$. Now we can use the fact that

$$\mathrm{cd}_H(K(J, J, H); M) \leq m - 1 + \log_p|H|.$$

If H'' is a subgroup of $H \cap NJ$, then it follows from the definition of the cohomology groups that

$$H^n_{\{H''\}}(K(e, J, H); M) = H^n_{\{H''J\}}(K(e, J, H); M).$$

Proposition 2.6 implies that, for $J \subseteq H$, $H'' \subseteq H \cap NJ$, and $n \geq m + \log_p|H|$,

$$H^n_{\{H''\}}(K(e, J, H); M) = 0.$$

Hence, by Lemma 3.2(i), $H^n_{\{H''\}}(K^J, M) = 0$, whenever $J \subseteq H$, $H'' \subseteq H \cap NJ$, and $n \geq m + \log_p|H|$.

Thus $\mathrm{cd}(H \cap NJ, K^J, M)$ is a finite number. From Corollary 2.4(ii) we obtain that

$$\mathrm{cd}(H \cap NJ, K^J, M) = \mathrm{cd}(e, K^J, M) \leq m - 1$$

and this fact ends the proof. $\square$

Proof of Theorem 2.9. Let $\mathrm{cd}_{E_G(K)}(K, M) = d$ where d is a natural number. It follows from Theorem 2.8 that, for every p-group $H \in E_G(K)$ and every $J \subseteq H$, $\mathrm{cd}(H \cap NJ, K^J, M) \le d$. We will prove that, for every p-subgroup H of G,

$$\mathrm{cd}(H \cap NJ, K(e, J, H); M) \le d + \log_p|H/J|$$

whenever $J \subseteq H$. We will proceed by the induction on $|H|$ and then on $|H/J|$. If $J = H$, then

$$K(e, J, J) = (K^J, \emptyset).$$

If J is not a normal subgroup of H, then we can use the induction on $|H|$ because

$$K(e, J, H) = K(e, J, H \cap NJ).$$

If, for every $k \in K$, $H_k \neq J$, then

$$\mathrm{cd}(H \cap NJ, K(e, J, H); M) = -1.$$

Assume now that J is a normal subgroup of H and that there is a point $k \in K$ such that $J = H_k$. Let J' be subgroup of H such that J is a normal subgroup of J' and J'/J is an elementary abelian p-group. It follows from the definition that $J' \in E_G(K)$. The inductive assumptions imply that

$$\mathrm{cd}_{J' \cap NH'}(K(e, H', H); M) \le d + \log_p|H/H'|$$

whenever $J \subseteq H' \subseteq H$ and $H' \neq J$. From the fact that $\mathrm{cd}_{J'}(K^J, M) \le d$ it follows that

$$\mathrm{cd}_{J'}(K(e, J, H); M) \le d + \log_p|H/J|.$$

From Proposition 2.6 we obtain that

$$\mathrm{cd}_{H''}(K(e, J, H); M) \le d + \log_p|H/J|$$

whenever $J \subseteq H'' \subseteq H$. If H' is a subgroup of H, then

$$\mathrm{cd}_{H'}(K(e, J, H); M) = \mathrm{cd}_{H'J}(K(e, J, H); M).$$

Thus

$$\mathrm{cd}(H, K(e, J, H); M) \le d + \log_p|H/J|.$$

In particular, for every Sylow p-subgroup G_p of G and $J \subseteq G_p$,

$$\mathrm{cd}(G_p \cap NJ, K(e, J, G_p); M) < \infty.$$

The result can be now obtained from Lemma 3.2(i) and Corollary 2.4. $\square$

In order to prove Proposition 3.3 we now state and prove Proposition 3.4 which is the main technical result of this paper. Proposition 3.4 is independent from the main results of [6] and [7].

Proposition 3.4. *Let H be a p-subgroup of G. Assume that J is a normal subgroup of H such that H/J is an elementary abelian p-group. Assume that*

$$H^n_{\{J'\}}(K, M) = 0$$

whenever $J \subseteq J' \subseteq H$ and $n = m + \log_p|J'/J|, m + 1 + \log_p|J'/J|$. Then

$$H^n_{\{J'\}}(K, M) = 0 = H^n_{\{H\}}(K(J', J', H), M)$$

whenever $J \subseteq J' \subseteq H$ and $m \leq n \leq m + 1 + \log_p|J'/J|$.

In order to prove the proposition above we need the following fact.

Lemma 3.5. *Let H be a subgroup of G and let J be a normal subgroup of H such that H/J is a cyclic p-group. Let $M_c[H/J]$ be the coefficient system defined on $O_{G,H}$ which is equal to the cokernel of the natural inclusion $\iota: M \to M[H/J]$. Assume that K' is an H-subcomplex of K such that, for every $k \in K'$, $H_k = J_k$. Then the following two conditions are equivalent:*

(i) $H^n_{\{H\}}(K', M) = 0 = H^n_{\{J\}}(K', M)$, *for* $n = m$, $m + 1$.

(ii) $H^n_{\{H\}}(K', M_c[H/J]) = 0$, *for* $n = m$, $m + 1$.

Proof. It follows from the definition that, after the restriction to the category $O_{F(K'),H}$, we have two exact sequences:

$$0 \to M \to M[H/J] \to M_c[H/J] \to 0,$$

$$0 \to M_c[H/J] \to M[H/J] \to M \to 0.$$

The inclusion $M \to M[H/J]$ is the composition of two natural inclusions $M \to M_c[H/J]$ and $M_c[H/J] \to M[H/J]$. The implication (i) $\Rightarrow$ (ii) is obvious. If the condition (ii) holds, then, for $n = m$, $m + 1$,

$$H^n_{\{H\}}(K', M) = H^n_{\{J\}}(K', M)$$

and there are epimorphisms

$$H^n_{\{H\}}(K', M) \to H^n_{\{J\}}(K', M),$$

whose images are equal to 0. $\square$

Proof of Proposition 3.4. We will use the induction on $|H|$ and then on $|H/J|$. Assume first that $H = J'$. In this case we have to prove that

$$H^n_{\{H\}}(K, M) = 0$$

whenever $m \leq n \leq m + 1 + \log_p|H/J|$. It is sufficient to consider only the case where $|H/J| = p$. The assumptions imply that, for $n = m, m + 1$,

$$H^n_{\{H\}}(K, M_c[H/J]) = 0.$$

It follows from the definition that

$$H^n_{\{H\}}(K, M_c[H/J]) = H^n_{\{H\}}(K(J, J, H); M_c[H/J]).$$

Lemma 3.5 implies that, for $n = m, m + 1$,

$$H^n_{\{H\}}(K(J, J, H); M) = 0 = H^n_{\{J\}}(K(J, J, H); M).$$

Let $K_0 = \{k \in K \mid H_k \neq J_k\}$. Then

$$H^n_{\{H\}}(K_0, M) = H^n_{\{J\}}(K_0, M) = 0.$$

Hence $H^n_{\{H\}}(K, M) = 0$ because $K(J, J, H) = (K, K_0)$. Thus the case $H = J'$ is proved.

We will need the following notation. Suppose that $J \subseteq H'' \subseteq H' \subseteq H$. Let

$$M\{H'', H'\} : O_{G,H} \to \mathrm{Ab}$$

be the contravariant functor such that, for $L \subseteq G$,

$$M\{H'', H'\}(G/L) = M(G/L)$$

in the case where $H' \cap L = H'' \cap L$ and is equal to 0 otherwise. If $f: G/L' \to G/L$ is a morphism of $O_{G,H}$ and $M\{H'', H'\}(G/L) \neq 0$, then $M\{H'', H'\}(f) = M(f)$. It is easy to check that

$$H^n_{\{H\}}(K, M\{H'', H'\}[H, H']) = H^n_{\{H'\}}(K(H'', H'', H'); M).$$

Assume now that $H \neq J'$. It follows from the assumptions that there exists a subgroup H' of H such that $J' \subseteq H'$ and $|H/H'| = p$. Let

$$\mathscr{A} = \{J'' \subseteq H \mid J'' \cap H' = J' \neq J''\}.$$

The set $\mathscr{A}$ is not empty. Suppose that $H'' \in \mathscr{A}$. Let $\mathscr{B} = \mathscr{A} \setminus \{H''\}$. Then there exists an exact sequence of functors

$$0 \to M\{H'', H\} \to M\{J', H'\}[H, H'] \to M' \to 0$$

where

$$M' = (M\{J', H\})_c[H, H'] \oplus \bigoplus_{J'' \in \mathscr{B}} M\{J', H\}/M\{J', H\}.$$

From the inductive hypotheses we obtain that

$$H^n_{\{H\}}(K(J', J', H); M_c[H, H']) = 0$$

whenever $m \leq n \leq m + 1 + \log_p|J'/J|$. Now it is sufficient to apply Lemma 3.5. □

Proof of Proposition 3.3. For any p-subgroup J of G, let $\phi(J)$ be the Frattini subgroup of J [3]. This means that $\phi(J)$ is the intersection of all maximal subgroups of J. If $\phi(J) \subseteq J' \subseteq J$, then J' is a normal subgroup of J and J/J' is an elementary abelian p-group. It follows from the definition that

$$K(\phi(J), J, J) = K(e, J, J) = (K^J, \emptyset).$$

Assume that, for $n \geq m$ and $J \subseteq H$, $H^n_{\{J\}}(K, M) = 0$. Proposition 3.4 implies that

$$H^n_{\{J\}}(K(J', J', J); M) = 0$$

whenever $n \geq m$ and $\phi(J) \subseteq J' \subseteq J \subseteq H$. By induction on $|J'/\phi(J)|$ it is easy to prove that

$$H^n_{\{J\}}(K(\phi(J), J', J); M) = 0$$

whenever $\phi(J) \subseteq J' \subseteq J$ and $n \geq m$. Thus

$$H^n_{\{e\}}(K^J, M) = H^n_{\{J\}}(K(\phi(J), J, J); M) = 0$$

whenever $n \geq m$ and $J \subseteq H$ and this ends the proof. □

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